Given $f(x,y)$

Def 1: the gradient of f

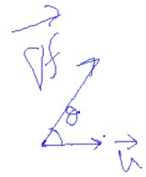
$$\overrightarrow{grad}(f) \equiv \overrightarrow{\nabla f} = f_x \vec{i} + f_y \vec{j}$$

Def 2: The derivative of f in the direction $\vec{u}$ is $D_u f = \overrightarrow{\nabla f} \cdot \vec{u}$

Remark $\vec{u}$ is called a direction vector if $|\vec{u}| = 1$

Def 3: If $\vec{v} \neq \vec{0}$, then the direction of $\vec{v}$ is $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$

$$D_u f = \overrightarrow{\nabla f} \cdot \vec{u} = |\overrightarrow{\nabla f}|\,|\vec{u}| \cos\theta$$



$$\therefore \quad -|\overrightarrow{\nabla f}| \le D_u f \le |\overrightarrow{\nabla f}|$$

<u>Rules:</u>

1. f increases most rapidly in the direction of $\overrightarrow{\nabla f}$. i.e. $\dfrac{\overrightarrow{\nabla f}}{|\overrightarrow{\nabla f}|}$. And the derivative of f in this direction is $|\overrightarrow{\nabla f}|$.

2. f decreases most rapidly in the direction $-\dfrac{\overrightarrow{\nabla f}}{|\overrightarrow{\nabla f}|}$ and the derivative of f in this direc. is $-|\overrightarrow{\nabla f}|$

3. We say that $\vec{u}$ is a direction of zero change for f when $\vec{u} \perp \overrightarrow{\nabla f}$.

$$\begin{array}{ccc} & f & \\ \swarrow & \downarrow & \searrow \\ x & & y \\ \searrow & \downarrow & \swarrow \\ & t & \end{array}$$

$$\nabla f = f_x \vec{i} + f_y \vec{j}$$

$$\boxed{C_k : \quad f(x,y) = k}$$

$$\frac{df}{dt} = \frac{dk}{dt}$$

$$\Downarrow$$

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

$$\vec{\nabla f} \cdot \vec{v} = 0$$

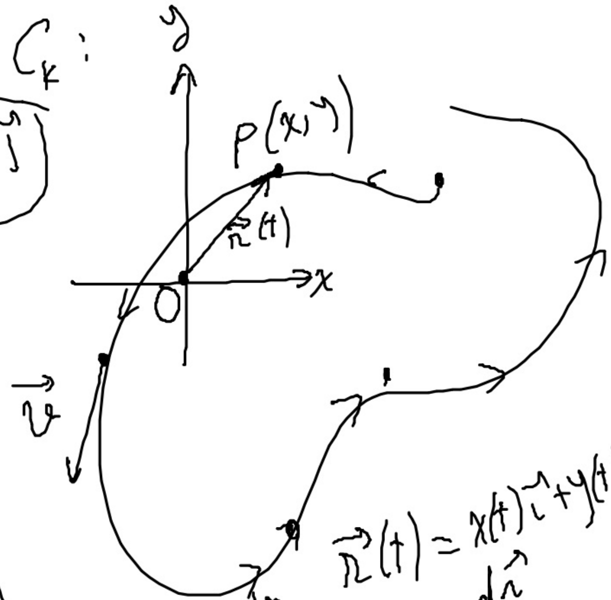
∇f

$\vec{\nabla f} \neq \vec{0}$

$\vec{v} \neq \vec{0}$

$$\Downarrow$$

$$\nabla f \perp \vec{v}$$



Position vector:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

$$\vec{v} \neq \vec{0}$$

ex4

$$\frac{x^2}{4}+\frac{y^2}{1}=2 \qquad P(-2,1)$$

$$\frac{x^2}{8}+\frac{y^2}{2}=1$$

let $f(x,y)=\frac{x^2}{4}+y^2$

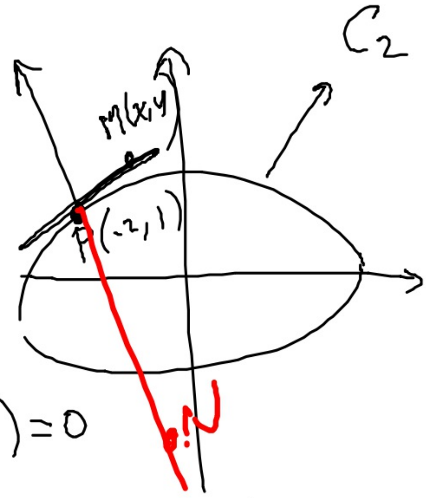
$$\vec{\nabla f}=\frac{x}{2}\vec{i}+2y\vec{j}$$

$$\boxed{\vec{\nabla f}(-2,1)=-\vec{i}+2\vec{j}}$$

let $M(x,y)$ be a pt on the tangent line

then $\vec{PM}\perp\vec{\nabla f}(P)$

i.e. $\vec{PM}\cdot\vec{\nabla f}(P)=0$



$\therefore -(x+2)+2(y-1)=0$

...

$$\vec{PM}=\left|\begin{matrix} x+2 \\ y-1 \end{matrix}\right.$$

#7 $f(x,y,z) = x^2+y^2-2z^2+z\ln x$; $(1,1,1)$

$f_x = 2x + \frac{z}{x}$; $f_x(1,1,1) = 3$

; $f_y(1,1,1) = 2$

$f_y = 2y$

; $f_z(1,1,1) = -4$

$f_z = -4z + \ln x$

$\therefore \vec{\nabla f}(1,1,1) = 3\vec{i} + 2\vec{j} - 4\vec{k}$.

Ⅳ $f(x,y,z) = xy + yz + zx$; $P_0(1,-1,2)$; $\vec{v} = 3\vec{i} + 6\vec{j} - 2\vec{k}$.

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}.$$

$$= \frac{3}{7} + \frac{18}{7} = 3.$$

Extra:

the direction of most rapid increase is the direction of $\vec{\nabla} f(P_0)$

i.e $\frac{1}{\sqrt{10}}(\vec{i} + 3\vec{j})$

$$|\vec{v}| = \sqrt{9+36+4} = 7$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}.$$

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

$$= (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

$$\vec{i} + 3\vec{j} = \vec{\nabla} f(P_0)$$

19 $f(x,y) = x^2 + xy + y^2$; $P(-1,1)$

$$\boxed{-|\vec{\nabla} f| \leq D_u f \leq |\vec{\nabla} f|}$$

$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} = (2x+y)\vec{i} + (x+2y)\vec{j}$

$\vec{\nabla} f(-1,1) = -\vec{i} + \vec{j}$.

- f increases most rapidly in the direction $\frac{-\vec{i}}{\sqrt{2}} + \frac{\vec{j}}{\sqrt{2}}$
  + the derivative of f in this direction is $\sqrt{2}$
- f decreases " " " " " $\frac{\vec{i}}{\sqrt{2}} - \frac{\vec{j}}{\sqrt{2}}$
  + the derivative of f in this direction is $-\sqrt{2}$

- The directions of zero change are $\frac{\vec{i}}{\sqrt{2}} + \frac{\vec{j}}{\sqrt{2}}$

  & $-\left(\frac{\vec{i}}{\sqrt{2}} + \frac{\vec{j}}{\sqrt{2}}\right)$

27 $xy = -4$

$P(2,-2)$

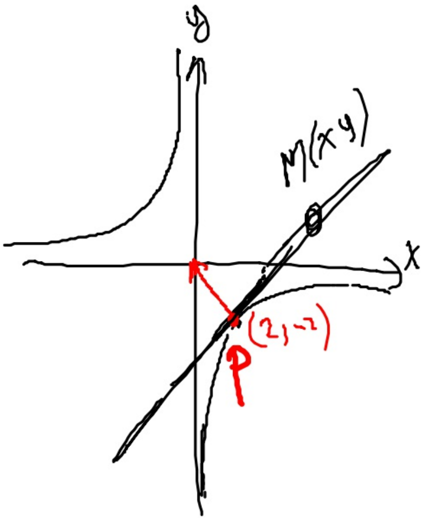
let $f(x,y) = xy$

then $xy = -4$

is C_{-4}

$\vec{\nabla f} = y\vec{i} + x\vec{j}$

$\vec{\nabla f}(2,-2) = -2\vec{i} + 2\vec{j}$



$\vec{PM} = \begin{vmatrix} x-2 \\ y+2 \end{vmatrix}$

let $M(x,y)$ be a pt on the tangent line, then

$\vec{PM} \perp \vec{\nabla f}(P)$

$\boxed{\vec{PM} = (x-2)\vec{i} + (y+2)\vec{j}}$

$-2(x-2) + 2(y+2) = 0$

33

$f(x,y) = x^2 - 3xy + 4y^2$; $P(1,2)$

solve : $D_u f(P) = 14$

$f_x = 2x - 3y$; $f_y = -3x + 8y$

No solutions

because $14 \notin [-\sqrt{185}, \sqrt{185}]$

Recall : $\boxed{-|\nabla f(P)| \leq D_u f(P) \leq |\vec{\nabla} f(P)|}$

However : $\vec{\nabla} f(P) = -4\vec{i} + 13\vec{j}$.

$|\vec{\nabla f}(P)| = \sqrt{185}$

$\simeq 13.6\ldots$

#35

Given: (a) the der. of f at (1,2) in the direction of $\vec{i}+\vec{j}$ is $2\sqrt{2}$.

& (b) " " " " " " " $-2\vec{j}$ is -3.

Find $\vec{\nabla} f(1,2)$. (Assume that $\vec{\nabla} f(1,2) = f_x \vec{i} + f_y \vec{j}$)

(a) $\Longrightarrow \left(f_x \vec{i} + f_y \vec{j}\right) \cdot \left(\dfrac{\vec{i}}{\sqrt{2}} + \dfrac{\vec{j}}{\sqrt{2}}\right) = 2\sqrt{2}$

$\therefore f_x + f_y = 4$ eq (1)

(b) $\left(f_x \vec{i} + f_y \vec{j}\right) \cdot \left(-\vec{j}\right) = -3$

$\therefore f_y = 3$

$\Longrightarrow \vec{\nabla} f(1,2) = \vec{i} + 3\vec{j}$.

36 $\vec{w} = \vec{i} + \vec{j} - \vec{k}$.

a) f increases most rapidly in the direction $\dfrac{\vec{\nabla} f(P)}{|\vec{\nabla} f(P)|}$

& the derivative of f in this direction is $|\vec{\nabla} f(P)|$

. Since f increases most rapidly in the direction of $\vec{w}$,

then $\dfrac{\vec{\nabla f(P)}}{|\vec{\nabla} f(P)|} = \dfrac{\vec{w}}{|\vec{w}|} = \dfrac{\vec{i}}{\sqrt{3}} + \dfrac{\vec{j}}{\sqrt{3}} - \dfrac{\vec{k}}{\sqrt{3}}$.

$\vec{A} = |\vec{A}| \dfrac{\vec{A}}{|\vec{A}|}$

. Since the derivative of f in the direction of $\vec{w}$ is $2\sqrt{3}$

then $|\vec{\nabla} f(P)| = 2\sqrt{3}$. $\Longrightarrow$

$\vec{\nabla f}(P) = 2\vec{i} + 2\vec{j} - 2\vec{k}$.

b) derivative of f in the direction of $\vec{i}+\vec{j}$ is

$$\vec{\nabla} f(P) \cdot \left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right) =$$

$$\left(2\vec{i}+2\vec{j}-2\vec{k}\right) \cdot \left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(2+2\right) = 2\sqrt{2}.$$

Given a vector $\vec{v} = a\vec{i} + b\vec{j}$

find a vector $\vec{w}$ s.t. $\vec{v} \perp \vec{w}$

& $|\vec{v}| = |\vec{w}|$.

<u>Ans</u>:

$\vec{w} = -b\vec{i} + a\vec{j}$

or $b\vec{i} - a\vec{j}$